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niveau lines at intervals of 1 c.m. were designated. A rather more elaborate mechanism is necessary to mark the lines of flow $v=\text{constant}$. These lines are so drawn that their vertical projections on the x, y plane are orthogonal to the niveau curves $u=\text{constant}$.

The model, thus constructed, was sent to Hrn. Kreittmayer in Munich, for duplication. During the summer of 1899 I worked for several days in his workshops but didn't learn enough to make a complete report of this interesting process.

The elliptic integral of the second kind is defined as the logarithmic derivative of the sigma function. The procedure for this function was essentially the same as for the sigma function, but here four parallelograms of periods are sufficient, as the function is periodic with regard to one of the periods of the elliptic function. The three theorems for sigma still hold for this function.

One fact is at once apparent from the model of the sigma function; for values of z within the first parallelogram it is of practical value in numerical calculations, but comparatively cumbersome for the next parallelogram and utterly worthless for parallelograms still further removed.

Both models are published by Martin Schilling, in Halle, successor to S. Brill, in Darmstadt.

MERIDIAN AND TRANSVERSE SECTIONS OF HELICOIDS OF UNIFORM PITCH.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

The following discussion was suggested by the treatment of this subject given in MacCord's Descriptive Geometry. In addition to establishing analytically the facts stated in that text-book, other properties of the curves of intersection have been determined.

The helicoid here considered is a surface generated by a straight line always tangent to a right circular cylinder, inclined at a constant angle to the axis of the cylinder, and having two simultaneous uniform motions—one of revolution about the axis, the other of translation parallel thereto. The locus of the point of tangency of this generatrix with the cylindrical surface is a common helix.

Let AB be the axis of the cylinder, C the center of a right section which intersects the helix OH at O , Q any point on OH , RT the corresponding position of the generatrix piercing the plane ABS , determined by the axis and O , at P , and meeting the plane of the right section at T .

Then P is any point of the meridian section of the helicoid and T any point of the transverse section.

A. TO FIND THE LOCUS OF P .

The plane through the generatrix RT and tangent to the cylinder cuts CO produced at E , ET being tangent to the circular section at D .

PE is parallel to QD which is an element of the cylindrical surface.

Draw through Q a parallel to DE , meeting PE at L .

Draw CO and CD .

Denote the angle COD by θ , the radius of the cylinder by a , the pitch of the helix by P , and the angle PTE by α .

Taking CO produced as the X -axis, and OY , parallel to AB , as the Y -axis, $EO=x$, and $PE=y$.

$$x = EC - OC = a(\sec\theta - 1) \dots (1).$$

$$y = PL + QD.$$

$PL = LQ \tan \alpha = ED \tan \alpha = a \tan \theta \tan \alpha = \text{catan} \theta$, letting $\tan \alpha = c$.

$$QD : P = \theta : 2\pi; \quad QD = \frac{\theta}{2\pi} P = K\theta,$$

where $K = \frac{P}{2\pi}$.

$$\text{Therefore, } y = K\theta + \text{catan} \theta \dots (2).$$

(1) and (2) are the equations of the meridian section of the helicoid.

There are certain properties of this curve independent of the value of c . These are deduced first. Afterward, the variations in form and the singularities are noted as c varies from $+\infty$ to $-\infty$.

Changing the sign of θ does not affect x but gives to y values, equal numerically, but of contrary sign. Hence, the curve is symmetrical with reference to the X -axis.

Since when $\theta=0$, $y=x=0$, the curve passes through the origin.

When $\theta=\pm\frac{1}{2}\pi$, $x=\infty$, and y becomes infinite except when $c=0$. There are, therefore, infinite branches.

$$\frac{dy}{dx} = \frac{K + c \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{K \cos^2 \theta + ca}{a \sin \theta} \dots (3).$$

When $\theta=\frac{1}{2}\pi$, $\frac{dy}{dx}=c \dots (4)$, and the Y -intercept of the corresponding tangent $= \frac{\pi K}{2} + ca$.

An asymptote is $y=cx + \frac{1}{2}P + ca \dots (5)$, a line crossing AB at a distance

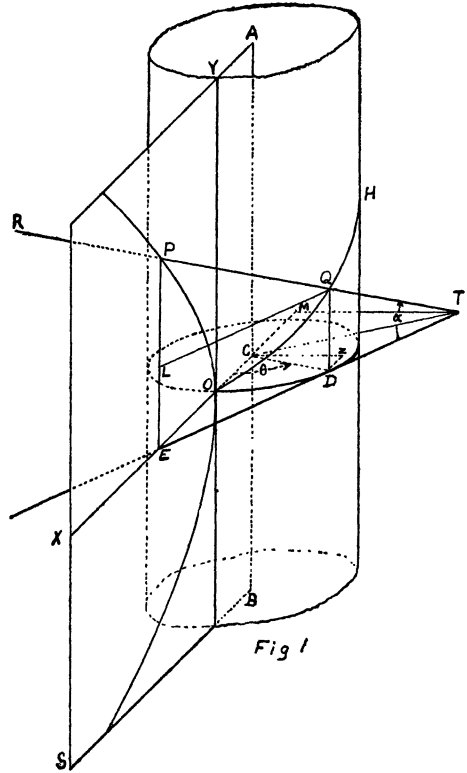


Fig 1

above C equal to one-fourth the pitch and making an angle with the X -axis equal to α .

Equating to zero, $\frac{1}{a}(K\cos^2\theta - 2K - ca)\cot\theta\csc\theta$, the derivative of $\frac{dy}{dx}$ with regard to θ , gives

$$\cos\theta = \pm \sqrt{2 + \frac{ca}{K}} \dots (6).$$

(a). $c > 0$.

From (3), when $\theta = 0$, $\frac{dy}{dx} = \infty$; the curve is tangent to the Y -axis at the origin.

From (6), there is no point of inflexion.

The curve is represented in Fig. 2, positive values of θ giving points above the axis on OP ; negative, points below on OW . UV is an asymptote, CU being $\frac{1}{4}P$.

(b). $c = 0$.

The equation of the asymptote becomes $y = \frac{1}{4}P$, the curve approaching parallelism with the X -axis, and appearing as indicated in Fig. 2 by the accented letters.

(c). $c < 0$.

Case I. $c < \frac{K}{a}$, numerically.

It follows from (2) that there is some X positive value of θ , less than $\frac{1}{2}\pi$ and other than zero, for which $y = 0$. At this point

$$\frac{dy}{dx} = \frac{c}{2} \frac{2\theta - \sin 2\theta}{\theta \cdot \sin \theta},$$

which is positive when θ is negative, and negative when θ is positive. Hence the point under consideration is a double point of intersection.

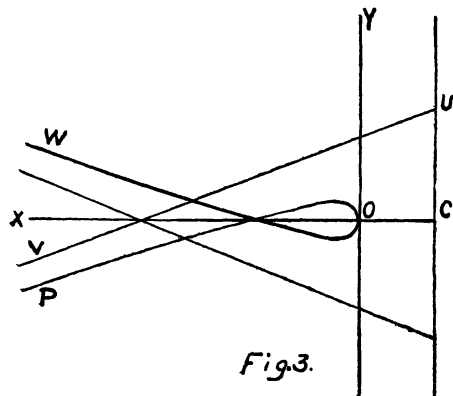


Fig. 3.

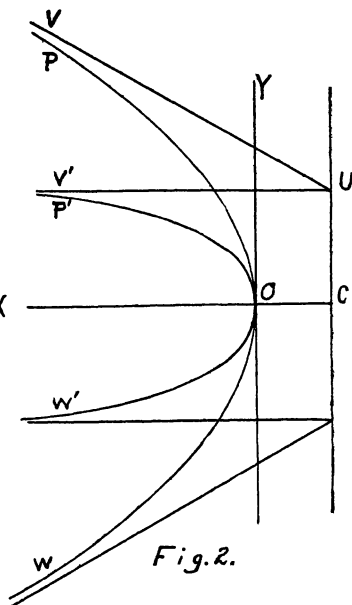


Fig. 2.

Somewhere between it and the origin the curve is parallel to the X -axis, this being

the case when $\theta = \cos^{-1} \sqrt{-\frac{ca}{K}}$.

(6) shows that there is no point of inflexion. See Fig. 3.

Case II. $c = \frac{K}{a}$, numerically.

Since $K = -ca$, (2) becomes $y = K(\theta - \tan\theta)$, from which y is negative as θ increases from 0 to $\frac{1}{2}\pi$, and decreases without limit, having neither a maximum nor a minimum value for any value of θ .

When $\theta=0$, (3) assumes the form $\frac{0}{0}$. But it may be written

$$\frac{K(\cos^2\theta-1)}{a\sin\theta} (= -\frac{K}{a}\sin\theta),$$

which equals zero when $\theta=0$. Hence there is a cusp of the first species at the origin.

(6) gives $\cos\theta=\pm 1$, but this does not correspond to a point of inflexion since $\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$ does not change sign as θ passes through 0.

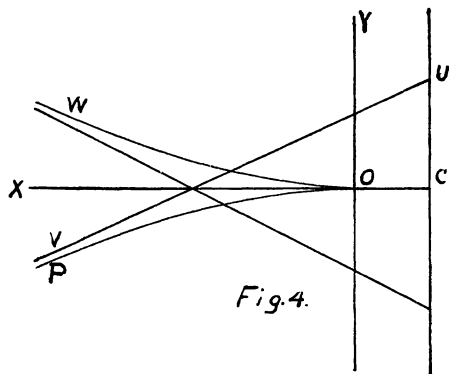
This curve is drawn in Fig. 4.

Case III. $c > \frac{K}{a}$ but $< \frac{2K}{a}$, numerically.

By (2) y is negative and θ increases from 0 to $\frac{1}{2}\pi$.

By (3) there is no point at which the curve is parallel to the X -axis; at the origin it is tangent to the Y -axis.

By (6) there is a point of inflexion where



$$\theta = \cos^{-1} \sqrt{2 + \frac{ca}{K}}.$$

See Fig. 5.

Case IV. $c > \frac{2K}{a}$, numerically, but negative.

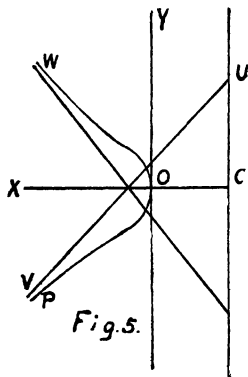
Points corresponding to θ between 0 and $\frac{1}{2}\pi$ lie below the X -axis. The curve is tangent to the Y -axis at the origin, and, by (6), there is no point of inflexion.

The curve is shown in Fig. 6.

When $c=\infty$, the helicoidal surface becomes the cylindrical, the meridian section coinciding with the Y -axis. As c changes sign, passing through infinity, the portions of the curve, OP and OW , exchange positions with reference to the X -axis.

It may be noted that in (c) Case II the inclination of the generatrix equals that of the helix, the helicoid becoming the convolute.

Equation (4) shows that the curve at infinity always has the direction of the generatrix when parallel to the meridian plane. This plane remaining fixed, while the inclination of the generatrix varies, the asymptote turns about the stationary point U through 180° , as is evident from equation (5).



B. TO FIND THE LOCUS OF T .

Through T draw a line perpendicular to OC , meeting it, produced, at M . Through C draw a parallel to MT meeting a perpendicular to it from D at Z .

Taking CO and CZ for the X - and Y -axes, respectively, $MC=x$ and $MT=y$.

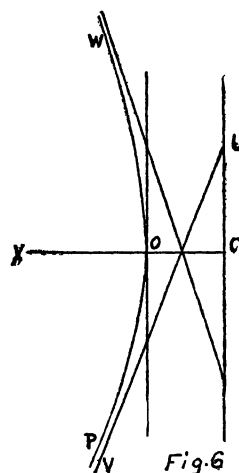
$$\text{Since } DT = \frac{QD}{\tan \alpha} = \frac{K\theta}{c},$$

$$x = a \cos \theta - \frac{K\theta}{c} \sin \theta \dots (1),$$

$$\text{and } y = a \sin \theta + \frac{K\theta}{c} \cos \theta \dots (2),$$

from which

$$x^2 + y^2 = a^2 + \frac{K^2 \theta^2}{c^2}, \text{ or, } r^2 = a^2 + \frac{K^2 \theta^2}{c^2}, (r = CT) \dots (3).$$



These are the equations of the transverse section. They show—(1) and (2) together, or (3) alone—that the curve is symmetrical with respect to the X -axis; it is a spiral beginning at O and cutting the axes of X and Y , in general, an infinite number of times at points more and more remote from C .

As in the case of the meridian section, this locus has distinguishing features depending on the value of C . These are shown in Fig. 7, a part of the first spiral only being represented. The loci are traced for values of θ between 0 and π ; also, symmetrical curves, OX being the axis of symmetry, for numerically equal negative values of θ .

$$\frac{dy}{dx} = \frac{K\theta \tan \theta - (ca + K)}{K\theta + (ca + K) \tan \theta} = \tan(\theta - \phi) \dots (4)$$

$$\text{where } \phi = \tan^{-1} \frac{ca + K}{K\theta}.$$

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \sec^2(\theta - \phi) \frac{K^2 \theta^2 + c^2 a^2 + 3caK + 2K^2}{K^2 \theta^2 + (ca + K)^2} \dots (5).$$

(a). $c > 0$.

The curve is perpendicular to the X -axis at O , is concave toward C , and has no point of inflexion. In Fig. 7, OTI is a part of a spiral of this class.

(b). $c = 0$.

The transverse section coincides with the generatrix when $\theta = 0$. In Fig. 7, it is shown as OY' , parallel to the Y -axis.

(c). $c < 0$.

Case I. $c < \frac{K}{a}$, numerically.

While the X -axis is still crossed at right angles at O , y is negative until θ reaches a value given by the equation

$$\tan \theta + \frac{K}{ca} \theta = 0.$$

By (4), the slope is first negative and, by (5) it increases with θ .

Hence, there is a double point of intersection as seen in $OT2$.

Case II. $c = \frac{K}{a}$, numerically.

(3) reduces to $r^2 = a^2 + a^2 \theta^2$, the transverse section becoming the involute of the circular section.

(4) gives $\frac{dy}{dx} = \tan \theta$,

as might have been inferred from the relation between the curve and its evolute.

When $\theta = 0$, $\frac{dy}{dx} = 0$, and there is a cusp of the first species at O . There being no point of inflexion, the locus appears as in $OT3$.

Case III. $c > \frac{K}{a}$ but $< \frac{2K}{a}$, numerically.

Again the X -axis is intersected perpendicularly at O .

Equating (5) to zero,

$$\theta = \sqrt{-\left(\frac{ca}{K} + 1\right)\left(\frac{ca}{K} + 2\right)}.$$

Hence there is a point of inflexion. See $OT4$.

Case IV. $c > \frac{2K}{a}$, numerically, but negative.

There is no point of inflexion. The curve is perpendicular to the X -axis at O and is concave toward C . $OT5$ represents it.

When $c = \infty$, (1) and (2) become $x = a \cos \theta$ and $y = a \sin \theta$, respectively; while (3) reduces to $K^2 + y^2 = a^2$, or, $r = a$. The transverse section coincides with the right circular section. See $OT6$.

The curve intersects the line OY' for some value of θ less than π when c lies between 0 and $-\frac{2K}{a}$. That this is true may be seen by making $x = a$ in (1), obtaining, after reduction,

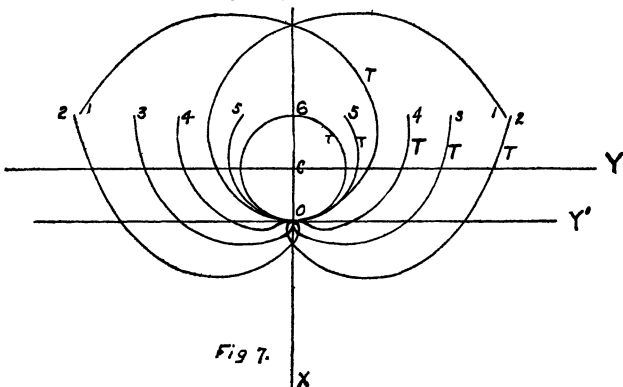


Fig 7.

$$\tan \frac{1}{2}\theta = -\frac{K'}{ca} \theta.$$

If $a=0$, (3) becomes $r = \pm \frac{K}{c}\theta$, the equation of a spiral of Archimedes.

Under this hypothesis, O and D (Fig. 1) coincide with C .

CT , coinciding with DT , is perpendicular to CD , and, since OCZ is a right angle, $\angle TCZ = \angle OCD = \theta$.

Hence CZ is the initial line and C the pole.

PROVING THE FALSE.

By DR. GEORGE BRUCE HALSTED.

I. THEOREM. Every triangle is isosceles.

HYPOTHESIS. Let ABC be a triangle.

CONCLUSION. $AC=BC$.

PROOF. Take the bisector of $\angle C$, and erect the perpendicular bisector of AB .

From their intersection M drop the perpendiculars MF , MH . Then $\triangle AMD \equiv \triangle BMD$ (by Euclid I. 4).

$\therefore AM=BM$.

But $\triangle CMH \equiv \triangle CMF$ (by Euclid I. 26). $\therefore MF=MH$, and $CF=CH$.

$\therefore \triangle AMH \equiv \triangle BMF$ (by Halsted's *Elements*, 179).

$\therefore BF=AH$. But already we have $FC=HC$.

Therefore $BC=AC$. Q. E. D.

II. Just as false is the following from Olney's *Geometry*, Section VIII, Proposition XIV.

THEOREM. "Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures."

A simple construction, due to my pupil R. L. Moore, shows that they may be as *unequal* as we please.

Make DG greater than DC by whatever sect you please, say a centimeter. Take $\angle DGF = \angle DCB$. Take $GF=CB$. Take A on the perpendicular bisector of BF . Then the quadrilaterals $ABCD$, $AFGD$ fulfill the hypothesis of the theorem.

My friend G. B. M. Zerr, in a moment of distraction, has given the following as proof that these unequals are equal, and it seems that Mr. J. Scheffer agrees therein. See THE AMERICAN MATHEMATICAL MONTHLY, April, 1902, Vol. IX, page 105.

